Concurrent Complementary Operators for Mesh Truncation in Frequency-Domain Simulations

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***Abstract*—The concurrent complementary operators method (C-COM), when used in the finite-difference time-domain method (FDTD), achieved levels of accuracy unprecedented in any previous mesh-truncation technique. In this paper, we introduce the extension of the C-COM theory to frequency-domain (time-harmonic) simulations. A validation of the new method is demonstrated by using the finite-difference frequency domain (FDFD) method.**

***Index Terms*—Absorbing boundary conditions, C-COM, complementary operators, FDFD.**

I. Complementary Operators

THE IDEA behind the complementary operators method (COM) is simple. Let us consider an outer boundary that is normal to the x axis in the Cartesian coordinates system. Two auxiliary differential operators, ∂_x and ∂_t, are applied to an absorbing boundary condition (ABC) such as Higdon, Liao, … etc. The purpose of these two auxiliary operators is to generate reflection coefficients that are precisely 180° out of phase, not only in the *analytic* domain, but also in the *discrete*, or numerical, domain. By averaging the solutions obtained from the application of each of the two operators on an ABC, we arrive at a new solution that is devoid of first-order reflections [1], [2].

The COM offers several advantages over other mesh-truncation techniques such as the perfectly matched layer (PML). When used in time-domain techniques, the COM and its subsequent variant, the concurrent complementary operators method (C-COM), were demonstrated to achieve higher accuracy than optimized PML [3]. In frequency-domain methods, the COM achieved unprecedented accuracy without the need for any *a priori* optimization, which is typically required for PML implementation [4], [5].

In this work, the C-COM is developed for frequency-domain (time-harmonic) methods. Although the frequency-domain and time-domain simulation are two different mathematical representations of the same physical phenomenon, the numerical solution paradigm is not expected to be identical. For this reason, complementary operators need to be numerically adapted to fit the particular numerical model under consideration [5].

Let us consider a computational boundary normal to the x axis and located at $x = a$. Furthermore, we assume that the problem domain is in $x < a$. When the complementary operators are applied to this boundary in a frequency-domain model, the resultant two discrete-domain complementary boundary operators are given by

$$D_x B = \frac{I - S^{-1}}{\Delta x} B \tag{1}$$

$$\overline{D}_x B = \frac{I + S^{-1}}{\Delta x} B \tag{2}$$

where I and S^{-1} are the identity and space shift discrete operators, respectively, and Δx is the grid spacing [5]. The corresponding reflection coefficients are given, respectively, as

$$R\{D_x B\} = -e^{jk_x \Delta x} R\{B\} \tag{3}$$

$$R\left\{\overline{D}_x B\right\} = e^{jk_x \Delta x} R\{B\} \tag{4}$$

where $R\{B\}$ is the reflection coefficient of the operator B and k_x is the wave number in the x direction. Notice that $R\{D_x B\}$ and $R\{\overline{D}_x B\}$ are precisely 180° out of phase and, hence, full complementariness is achieved.

II. C-COM in Frequency-Domain Algorithms

Let us consider the problem of TM-polarized radiation in two-dimensional space. The governing equation is the Helmholz wave equation given by

$$\nabla^2 E_z + k^2 E_z = 0. \tag{5}$$

For clarity, we limit the discussion here to the finite-difference frequency-domain (FDFD) method (the implementation and application in the finite element method is similar). The first step in the implementation of the FDFD method is to divide the computational region into grid. Following a similar procedure used in the implementation of the C-COM in time-domain simulation [2], we divide the computational domain (grid) into a boundary region (or layer) and an interior region as shown in Fig. 1. The width of the boundary layer must be equal or greater than the stencil needed to implement the operators in (1) and (2) [2]. To each field node in the boundary layer, we assign two field values, E_{z1} and E_{z2}. In the interior region, we assign a single field value, E_z to each node, as in conventional implementation. Next, we apply the second-order accurate finite-difference scheme to each node in the interior region. For ease of illustration, we assume that the grid is uniform in the x and y directions, and let Δ be the grid spacing.

Applying the FDFD scheme to the free-space Helmholtz equation at an interior node $(i, j)_i$, we have

$$E_z(i-1, j)_i + E_z(i+1, j)_i + E_z(i, j+1)_i + E_z(i, j-1)_i + (k^2\Delta^2 - 4)E_z(i, j)_i = 0. \tag{6}$$

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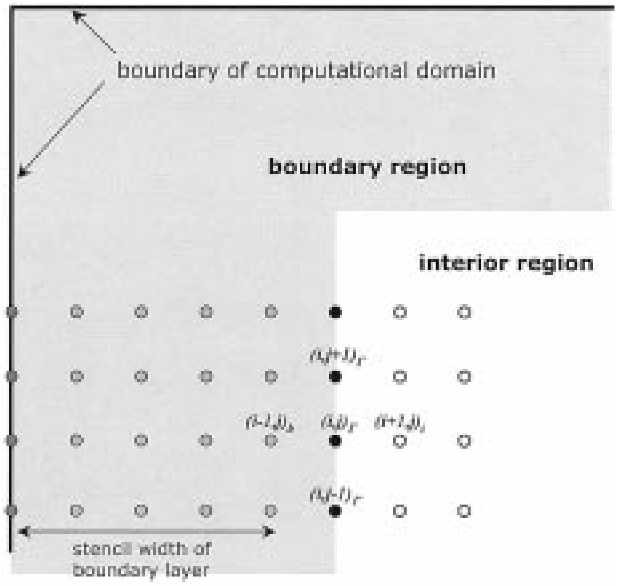

Fig. 1. Computational domain for the FDFD method showing grid used for the application of the C-COM mesh truncation technique.

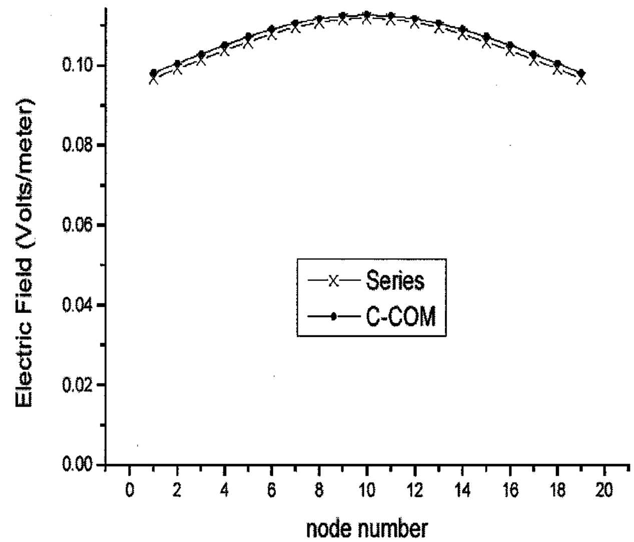

Fig. 2. The E_z field computed across the top of a $21\Delta \times 21\Delta$ computational domain using FDFD with C-COM applied for mesh truncation (C-COM) and the Hankel function series solution (Series).

In the boundary region, we apply the finite-difference equation to each set of fields designated by the subscripts $z1$ and $z2$. Designating the boundary nodes by the b subscript, we have

$$E_{z1}(i-1,\,j)_b + E_{z1}(i+1,\,j)_b + E_{z1}(i,\,j+1)_b + E_{z1}(i,\,j-1)_b + (k^2\Delta^2-4)E_{z1}(i,\,j)_b = 0 \quad (7)$$

$$E_{z2}(i-1,\,j)_b + E_{z2}(i+1,\,j)_b + E_{z2}(i,\,j+1)_b + E_{z2}(i,\,j-1)_b + (k^2\Delta^2-4)E_{z2}(i,\,j)_b = 0 \quad (8)$$

The next step is the implementation of the complementary operators. The first operator D_xB is applied to the set of fields denoted by E_{z1}, and the second operator $\overline{D}_xB$ is applied to the set of fields E_{z2}. Let Γ be the interface or perimeter between the boundary layer and the interior region, as illustrated in Fig. 1. The fundamental mechanism of the C-COM method is the averaging process needed to annul first-order reflections. To see how the averaging process is implemented, we focus on the left-hand-side segment of Γ. In Fig. 1, we show the grid on and in the close proximity of Γ. On Γ, the update equation for the fields uses the average field values $0.5(E_{z1}+E_{z2})$ from the left-hand side and E_z from the interior region (interior region includes the boundary layer Γ). For clarity, we designate the nodes on the boundary Γ by the subscript Γ. Thus, the finite-difference equation for the fields on Γ is given by

$$\frac{E_{z1}(i-1,j)_b + E_{z2}(i-1,j)_b}{2} + E_z(i+1,\,j)_i + E_z(i,\,j+1)_\Gamma + E_z(i,\,j-1)_\Gamma + (k^2\Delta^2-4)E_z(i,\,j)_\Gamma = 0. \quad (9)$$

Similar equations are applied on the other three sides of Γ. Notice that the averaging process expressed by (9) in addition to the increased storage requirements dictated by the C-COM implementation do not alter the sparsity structure of the final system matrix. In fact, the efficiency by which the new system matrix is solved is identical to the efficiency of the solution of typical FDFD matrix systems with or without the C-COM operations.

The implementation of (7)–(9) in an FDFD code does not require any extraordinary treatment. Assuming a square computational domain of N rows $\times N$ columns and a boundary layer of width W, the matrix size will be larger by $4NW - 4W^2$ in comparison to the case when C-COM is not implemented. For example, if the width of the boundary layer is $W = 5$ and the size of the domain is 100×100, i.e., $N = 100$, the increase in the matrix size will be 19% in comparison to the case where C-COM is not applied. For the case of $N = 1000$, the increase will be a marginal 2%.

The above presentation focused on the two-dimensional domain. The extension to three-dimensional problems is identical.

III. Numerical Validation

We consider the problem of a TM-polarized point source radiating in two-dimensional free space. The source is placed at the center of a $21\Delta \times 21\Delta$ computational domain. The grid spacing is $\Delta = 0.05\lambda$. The boundary layer is taken to be 5 cells wide ($W = 5$). We use Higdon's third-order boundary condition for B in (1) and (2). Higdon's boundary operators are well suited for Cartesian outer boundaries and thus are ideal for the finite-difference method. Fig. 2 shows comparison between the C-COM solution and the exact solution obtained from the analytical Hankel function series solution for the point source. The solution presented in Fig. 2 corresponds to the electric field on nodes lying across the top of the computational domain Notice that despite the close proximity of the terminal boundary to the source (0.5λ), the COM solution is observed to compare very favorably with the exact solution.

One should note that the wider the boundary layer, the higher the accuracy for fields observed within the interior region. This is because the absorption of evanescent waves increases with the position of the averaging layer with respect to the domain boundary [2].

IV. Conclusion

This work presented the extension of the C-COM theory for mesh truncation to frequency-domain (time-harmonic) simulation. The extension was made possible by implementing the concurrent averaging as a matrix operation. The application procedure was presented for the two-dimensional FDFD method. The extension to three-dimensional simulation is identical. A numerical experiment was presented validating the model and proving the effectiveness of this new mesh-truncation procedure. Application of the C-COM to other frequency-domain simulation methods such as the finite element method follows similar procedure and will be the subject of future work.

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